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## *A Hierarchical Bayesian approach for modeling the evolution of the 7-day moving average of the number of deaths by COVID-19*

E.F. Saraiva<sup>a\*</sup>, L. Sauer<sup>b</sup>, and C.A.B. Pereira<sup>c</sup>

<sup>a</sup>*Instituto de Matemática, Universidade Federal de Mato Grosso do Sul, MS, Brazil;*

<sup>b</sup>*Escola de Administração e Negócios, Universidade Federal de Mato Grosso do Sul, SP, Brazil;*

<sup>c</sup>*Instituto de Matemática e Estatística, Universidade de São Paulo, SP, Brazil,*

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In this paper, we propose a hierarchical Bayesian approach for modeling the evolution of the 7-day moving average for the number of deaths due to COVID-19 in a country, state or city. The proposed approach is based on a Gaussian process regression model. The main advantage of this model is that it assumes that a non-linear function  $f$  used for modeling the observed data is an unknown random parameter in opposite to usual approaches that set up  $f$  as being a known mathematical function. This assumption allows the development of a Bayesian approach with a Gaussian process prior for  $f$ . In addition, we assume one more hierarchical level, putting prior distributions on the parameters of the Gaussian process and on the variance of the random error. In order to estimate the parameters of interest, we develop an MCMC algorithm based on the Metropolis-within-Gibbs sampling algorithm. We also present a procedure for making predictions. The proposed method is illustrated in a case study, in which, we model the 7-day moving average for the number of deaths recorded in the state of São Paulo, Brazil. Results obtained show that the proposed method is very effective in modeling and predicting the values of the 7-day moving average.

**Keywords:** COVID-19, Bayesian Approach, Gaussian process, Predictions, MCMC algorithm.

### 1. Introduction

Especially in the year 2020, many articles were published describing modeling procedures for the number of cases and/or deaths due to COVID-19 in many countries. The interest in this kind of modeling lies mainly in projections that these models may provide and consequently assist government agents in making decisions regarding the intensification of social isolation, the acquisition of hospital equipment, an increase in the number of intensive care units in hospitals, among others.

In general, the published works model the accumulated number of cases (or deaths) by using some non-linear growth model. For example, [2] considered a simple exponential growth model to analyze the initial phase of the epidemic of COVID-19 in Africa, [3] modeled the data from Philippines and Taiwan using the generalized logistic model, [4] applied the Richards growth model to the data collected in China, France, Germany, Iran, Italy, South Korea, and Spain, [5] calibrated the logistic growth model, the generalized logistic growth model, and the generalized Richards model for the number of cases recorded in China, among others.

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\*Corresponding author. Email: erlandson.saraiva@ufms.br

Since the growth in the cumulative number of cases and deaths by COVID-19 has, in general, presented a heterogeneous evolution over time, this implies that the adjustment of only one of these growth models may not be adequate to explain the entire study period. This heterogeneous evolution is due to the occurrence of more than one wave of the pandemic or due to the accuracy of the statistical reports on the number of cases and deaths recorded. For example, the number of cases may present a fast growth if the number of the diagnostic tests is increased. On the other hand, tests that need to be realized to include the death cases in mortality statistics may take some days. This implies that may occur a sub-notification followed by an over-notification. In addition, the number of cases and deaths may be sub-notified on the weekends and appear on the statistical reports only a few days after.

An alternative is modeling, for instance, the 7-day moving average in order to minimize the discrepancies that may be contained in the dataset. In this paper, we assume for the 7-day moving average dataset an additive model composed by a non-linear function  $f$  plus a random error  $\epsilon$ . However, in opposite to set up  $f$  as a known mathematical function, we assume that  $f$  is an unknown random parameter. In order to estimate it from the data, we adopt a Bayesian approach putting over the unknown non-linear function a Gaussian process prior. In addition, we assume a more hierarchical level putting prior distributions on the parameters of the Gaussian process.

In order to estimate parameters of interest, we develop an MCMC algorithm based on the Metropolis-within-Gibbs sampling algorithm. We also present an MCMC algorithm for making predictions. The proposed method is illustrated in a case study, in which, we model the 7-day moving average recorded in the state of São Paulo, Brazil. Results obtained show that the proposed method is very effective in modeling the values of the 7-day moving average for the number of deaths due to COVID-19. In the course of 481 days of the pandemic, we run the estimation procedure four times, on the 100th, 180th, 280th and 465th days. In this four analysis the mean square error of the fitted model was smaller than 0.05, indicating a very good performance of the proposed method.

The three main advantages of the proposed method are: 1. It is very flexible since it is not restricted to a parametric mathematical function; 2. Does not need the fit of a set of parametric models followed of the application of a model comparison procedure; 3. It is easy to be implemented computationally since the estimation procedure is based on the use of a Metropolis-within-Gibbs sampling algorithm.

Although, we develop the paper with focus on 7-day moving average for the number of deaths, the method also can be used for modeling the moving average for the number of cases. In addition, the method is not restricted to the 7-day moving average and a user may use it for modeling a  $d$ -day moving average dataset, for  $d > 1$ .

The remainder of the paper is organized as follows. In Section 2, we present the Bayesian approach for modeling the 7-day moving average for the number of deaths recorded in a country, state and/or city. In this section, we also describe the MCMC algorithms used for estimate the parameters of interest and make predictions. Section 3, present an application of the proposed method to a case study. Section 4 concludes the paper with the final remarks. Additional details are provided in the Appendix.

## 2. Bayesian model for 7-day Moving Average

Let  $D_t$  be the number of deaths by COVID-19 recorded in a country, state or city on the  $t$ -th day, for  $t = 0, \dots, N$ , where  $t = 0$  represents the day that the first death was recorded and  $N$  is the last day considered in the analysis. Consider  $Y_t = \sum_{m=t}^{t+6} D_m$  be the 7-day moving average of the number of deaths due to the covid-19, for  $t = 1, \dots, n = N - 6$ .

Assume that  $Y_t$  values are generated according to the following additive model

$$Y_t = f_t + \varepsilon_t \quad (1)$$

where  $f_t = f(t)$  is an unknown function and  $\varepsilon_t$  is a random error assumed as being generated from a normal distribution with mean 0 and variance  $\sigma^2$ ,  $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$ , for  $t = 1, \dots, n$ .

At this point, it is usual to complete the model (1) by setting up  $f_t$  as a known nonlinear mathematical function. However, there may be several nonlinear parametric models that can fit the observed data points equally well. Due to this, it is common to fit a set of candidate models and then choose the best model using some model selection criteria, such as AIC or BIC. That is, the analysis stays limited to the set of models previously chosen by a user. In addition, the complexity and/or flexibility of the parametric models considered is limited by the number of parameters in the model.

In order to give flexibility to the modeling and not to be restricted to a set of parametric models, hereafter we assume that the unknown nonlinear function is a parameter of interest. In order to estimate these parameters from an observed dataset, we assume a Bayesian approach. Thus, considere  $\mathbb{F}_{\mathbf{f}}$  be the set of all possible functions that can explain the data. Let  $\mathbb{P}_{\mathbf{f}}$  be a probability distribution defined over  $\mathbb{F}_{\mathbf{f}}$ , in a way that, a finite set  $\mathbf{f} = (f_1, \dots, f_n)$  of  $\mathbb{F}_{\mathbf{f}}$  follows a multivariate normal distribution with mean vector  $\mathbf{m}$  of dimension  $1 \times n$  and covariance  $\Sigma_{\mathbf{f}}$  of dimension  $n \times n$  and elements  $\sigma(t, t') = cov(f_t, f_{t'})$ , for  $t, t' = 1, \dots, n$ . In other words, we are assuming that *a priori*  $\mathbf{f} \sim \mathcal{N}_n(\mathbf{m}, \Sigma_{\mathbf{f}})$ , *i.e.*, a Gaussian process prior over  $\mathbf{f}$ , where  $\mathcal{N}_n(\cdot)$  represents the  $n$ -variate normal distribution.

Thus, setting up  $\mathbf{m} = \mathbf{0}$  in order to represent our no informative prior knowledge about the expected value of  $\mathbf{f}$  and letting  $\Sigma_{\mathbf{f}}$  be an unknown quantity, we propose the following hierarchical Bayesian model

$$\begin{aligned} \mathbf{Y} | \mathbf{f}, \sigma_{\varepsilon}^2 &\sim \mathcal{N}_n(\mathbf{f}, \sigma_{\varepsilon}^2 \mathbb{I}); \\ \mathbf{f} | \Sigma_{\mathbf{f}} &\sim \mathcal{N}_n(\mathbf{0}, \Sigma_{\mathbf{f}}) \\ \sigma_{\varepsilon}^2 &\sim \mathcal{IG}\left(\frac{a}{2}, \frac{b}{2}\right); \\ \Sigma_{\mathbf{f}} | \gamma, \sigma_{\mathbf{f}}^2, \nu &\sim \mathcal{IW}(\gamma, \sigma_{\mathbf{f}}^2 \mathbb{A}_{\nu}). \\ \sigma_{\mathbf{f}}^2 | c, d &\sim \mathcal{G}\left(\frac{c}{2}, \frac{d}{2}\right), \\ \nu | g, h &\sim \mathcal{G}(g, h) \end{aligned}$$

where  $\mathcal{IG}(\cdot)$  and  $\mathcal{G}(\cdot)$  represent the inverse gamma and gamma distributions, respectively, and  $\mathcal{IW}(\gamma, \sigma_{\mathbf{f}}^2 \mathbb{A}_{\nu})$  represents the inverse-Wishart distribution with parameters  $\gamma$  and  $\sigma_{\mathbf{f}}^2 \mathbb{A}_{\nu}$  is a matrix of dimension  $n \times n$  with elements  $k(t, t')$ , for  $t, t' = 1, \dots, n$ . Each term  $k(t, t')$  is calculated according to the squared exponential kernel, *i.e.*,

$$k(t, t') = \exp\left\{-\frac{(t - t')^2}{2\nu^2}\right\}. \quad (2)$$

We also assume that  $a, b, c, d, g$  and  $h$  are known hiperparameters. We set up all of them equals to 0.1 in order to get non-informative prior distributions.

Using Bayes theorem, the joint posterior distribution for  $\boldsymbol{\theta} = (\mathbf{f}, \sigma_{\varepsilon}^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2, \nu)$  is

$$\pi(\boldsymbol{\theta} | \mathbf{y}, \mathbf{t}) \propto L(\boldsymbol{\theta} | \mathbf{y}, \mathbf{t}) \pi(\mathbf{f} | \Sigma_{\mathbf{f}}) \pi(\sigma_{\varepsilon}^2 | a, b) \pi(\Sigma_{\mathbf{f}} | \gamma, \sigma_{\mathbf{f}}^2, \nu) \pi(\sigma_{\mathbf{f}}^2 | c, d) \pi(\nu | g, h) \quad (3)$$

where  $L(\boldsymbol{\theta}|\mathbf{y}, \mathbf{t}) \propto (\sigma_\epsilon)^{-n/2} \exp\left\{-\frac{1}{2}(\mathbf{y} - \mathbf{f})^T \Sigma_\epsilon^{-1}(\mathbf{y} - \mathbf{f})\right\}$  is the likelihood function from a  $n$ -variate normal distribution, for  $\Sigma_\epsilon = \sigma_\epsilon^2 \mathbb{I}$ , being  $\mathbb{I}$  is the identity matrix of dimension  $n \times n$  and the overwritten  $T$  represents the matrix transpose.

The conditional posterior distributions are given by

$$\mathbf{f}|\bullet \sim \mathcal{N}_n\left(\Sigma_\epsilon^{-1}(\Sigma_\epsilon^{-1} + \Sigma_{\mathbf{f}}^{-1})\mathbf{y}, (\Sigma_\epsilon^{-1} + \Sigma_{\mathbf{f}}^{-1})^{-1}\right) \quad (4)$$

$$\sigma_\epsilon^2|\bullet \sim \mathcal{IG}\left(\frac{a+n}{2}, \frac{b + (\mathbf{y} - \mathbf{f})^T(\mathbf{y} - \mathbf{f})}{2}\right) \quad (5)$$

$$\Sigma_{\mathbf{f}}|\bullet \sim \mathcal{IW}(\gamma + n + 1, \mathbf{f}\mathbf{f}^T + \sigma_{\mathbf{f}}^2 \mathbb{A}_\nu) \quad (6)$$

$$\sigma_{\mathbf{f}}^2|\bullet \sim \mathcal{G}\left(\frac{c+n\gamma}{2}, \frac{d + \text{tr}(\mathbb{A}_\nu \Sigma_{\mathbf{f}}^{-1})}{2}\right), \quad (7)$$

$$\nu|\bullet \sim \pi(\nu|\bullet) \propto \pi(\Sigma_{\mathbf{f}}|\bullet)\pi(\nu|g, h), \quad (8)$$

where the symbol  $\bullet$  represents all other parameters.

Since Equations (4)-(7) have a known form and Equation (8) does not, then in order to get estimates for the parameters of interest, we consider a Metropolis-within-Gibbs sampling algorithm (MWGS). To each iteration of the MWGS algorithm, we update the parameters  $\boldsymbol{\theta}_{-\nu} = (\mathbf{f}, \sigma_\epsilon^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2)$  according to the algorithm 1.

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**Algorithm 1** Gibbs sampling algorithm

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- 1: Let the state of the Markov chain consist of  $\boldsymbol{\theta} = (\mathbf{f}, \sigma_\epsilon^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2, \nu)$ .
  - 2: Initialize the algorithm with a configuration  $\boldsymbol{\theta}^{(0)} = (\mathbf{f}^{(0)}, \sigma_\epsilon^{2(0)}, \Sigma_{\mathbf{f}}^{(0)}, \sigma_{\mathbf{f}}^{2(0)}, \nu^{(0)})$ .
  - 3: **procedure** For the  $l$ -th iteration of the algorithm,  $l = 1, \dots, L$
  - 4:   generate  $\mathbf{f}^{(l)}$  from conditional distribution (4), given  $\sigma_\epsilon^{2(l-1)}, \Sigma_{\mathbf{f}}^{(l-1)}$ ;
  - 5:   generate  $\sigma_\epsilon^{2(l)}$  from conditional distribution (5), given  $\mathbf{f}^{(l)}$ ;
  - 6:   generate  $\Sigma_{\mathbf{f}}^{(l)}$  from conditional distribution (6), given  $\mathbf{f}^{(l)}$  and  $\sigma_{\mathbf{f}}^{2(l-1)}$ ;
  - 7:   generate  $\sigma_{\mathbf{f}}^{2(l)}$  from conditional distribution (7), given  $\Sigma_{\mathbf{f}}^{(l)}$
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In order to update parameter  $\nu$  via Metropolis-Hastings algorithm, let  $\nu^*$  to be a candidate value generated from a candidate generating-density  $q[\nu^*|\nu]$ . So, the value  $\nu^*$  is accepted with probability  $\Psi(\nu^*|\nu) = \min(1, \mathbb{H}_\nu)$ , where

$$\mathbb{H}_\nu = \frac{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu^*) \pi(\nu^*|g, h) q[\nu|\nu^*]}{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu) \pi(\nu|g, g) q[\nu^*|\nu]}.$$

Now, it is necessary to specify the candidate-generating density  $q[\nu^*|\nu]$ . Usually  $q[\cdot]$  is chosen such that it is easy to sample from it. Two common choice are:

- $q[\nu^*|\nu] = \pi[\nu]$ , *i.e.*, the candidate generating-density is given by the prior distribution. In this case,  $\mathbb{H}_\nu$  simplifies to

$$\mathbb{H}_\nu = \frac{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu^*)}{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu)}.$$

This case is denominated in the literature by Independent Metropolis-Hastings (IMH). Although the choice of the prior distributions as the candidate generating-density is mathematically attractive, this may lead to many rejections of the proposed moves and a slow convergence of the algorithm. This happens, specially, for cases in which no prior information is available and prior distribution has large variance.

- An alternative to the IMH is to explore the neighbourhood of the current value of the chain in order to propose a new value. Thus, let  $q[\nu^*|\nu] = g(|\nu - \nu^*|)$ , where  $g(\cdot)$  is a symmetric density, *i.e.*, the probability of generating a move from  $\nu$  to  $\nu^*$  depends only on the distance between them. In this case,  $\mathbb{H}_\nu$  simplifies to

$$\mathbb{H}_\nu = \frac{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu^*) \pi(\nu^*|g, h)}{\pi(\Sigma_{\mathbf{f}}|\gamma, \sigma_{\mathbf{f}}^2, \nu) \pi(\nu|g, g)}. \quad (9)$$

This case is denominated in the literature by random walk Metropolis (RWM).

In the remaining of the paper, we adopt the RWM algorithm for updating  $\nu$ . But, this does not prevent the proposed method to be applied with the Metropolis step being given by the IMH. In order to implement a RWM, we set up  $\nu^* = \nu + \eta$  with  $\eta \sim \mathcal{N}(0, \sigma_\nu^2)$ . However, as discussed by [22] the choice of  $\sigma_\nu^2$  has great influence on efficiency of the algorithm. If  $\sigma_\nu^2$  is small, then random perturbations will be small in magnitude and almost all will be accepted, requiring a large number of iterations to get convergence. On the other hand, if  $\sigma_\nu^2$  is too large, then it will causes too many rejections of the proposed moves and a considerably slowing down convergence.

According to [23], [24], [25] and [26], one may fix the value of  $\sigma_\nu^2$  testing some values in a few pilot runs and then choosing a value in which acceptance ratio lies in between 20% and 30%. Thus, following this procedure we run 10 pilot runs of the algorithm with  $L = 10.000$  iterations for  $\sigma_\nu^2 \in \mathcal{G}$ , where  $\mathcal{G}$  is a grid from 0.1 to 1 with increments of size 0.1. For  $\sigma_\nu^2 = 0.1$  the acceptance rate was of 75%, for  $\sigma_\nu^2 = 0.2$  the acceptance rate was of 25.82%, for  $\sigma_\nu^2 = 0.3$  the acceptance rate was of 24.75% for  $\sigma_\nu^2 = 0.4$  the acceptance rate was of 21.12% and for the other values tested,  $\sigma_\nu^2 \geq 0.5$  the acceptance rate was smaller than 20%. Thus, we fix up  $\sigma_\nu^2 = 0.3$ , the mean of the tested values that have led to an acceptance rate between 20% and 30%.

The RWM for updating  $\nu$  is implemented according to the steps of the Algorithm 2. Using algorithms 1 and 2, we implemented the MWGS to get estimates for parameters  $\theta$ . The steps for the implementation of this algorithm are given in Algorithm 3.

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#### Algorithm 2 Random Walk Metropolis

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- 1: Let the state of the Markov chain consist of  $\theta = (\mathbf{f}, \sigma_\epsilon^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2, \nu)$ .
  - 2: Initialize the algorithm with a configuration  $\theta^{(0)} = (\mathbf{f}^{(0)}, \sigma_\epsilon^{2(0)}, \Sigma_{\mathbf{f}}^{(0)}, \sigma_{\mathbf{f}}^{2(0)}, \nu^{(0)})$ .
  - 3: **procedure** For the  $l$ -th iteration of the algorithm,  $l = 1, \dots, L$
  - 4:     Generate  $r \sim \mathcal{N}(0, \sigma_\nu^2)$  and set  $\nu^* = \nu(l-1) + r$
  - 5:     Calculate  $\Psi(\nu^*|\nu) = \min(1, \mathbb{H}_\nu)$ , where  $\mathbb{H}_\nu$  is given by Equation (9);
  - 6:     Generate  $u \sim \mathcal{U}(0, 1)$ . If  $u \leq \Psi(\nu^*|\nu)$  accept  $\nu^*$  and do  $\nu^{(l)} = \nu^*$ . Otherwise, reject  $\nu^*$  and set  $\nu^{(l)} = \nu^{(l-1)}$ .
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After running the  $L$  iterations of the algorithm 3, we discard the first  $B$  iterations as a burn-in. We also consider jumps of size  $J$ , *i.e.*, only 1 drawn from every  $J$  was extracted from the original sequence obtaining a sub sequence of size  $S = [(L - B)/J]$  to make inferences. The estimates for parameters of interest are given by average of the generated

values. For example, the estimates for  $\mathbf{f}$  and  $\sigma_\epsilon^2$  are  $\hat{\mathbf{f}} = \bar{\mathbf{f}} = (\bar{f}_1, \dots, \bar{f}_n)$  and  $\bar{\sigma}_\epsilon^2$ , where

$$\bar{f}(t) = \frac{1}{S} \sum_{l=1}^L f_t^{(M(l))}, \quad \hat{\sigma}_\epsilon^2 = \frac{1}{S} \sum_{l=1}^L \sigma_\epsilon^{2(M(l))}$$

and  $f_t^{(M(l))}$  and  $\sigma_\epsilon^{2(M(l))}$  are the generated values for  $f_t$  and  $\sigma_\epsilon^2$  in the  $M(l) = (B+1+l \cdot J)$ -th iteration of the algorithm, respectively, for  $t = 1, \dots, n$ . For the other parameters, the estimates are obtained in a similar way. The credibility interval (95%) for each one of the parameters is given by the quantiles 2.5% and 97.5% of the sampled values.

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**Algorithm 3** Metropolis-within-Gibbs algorithm (MWGS)

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- 1: Let the state of the Markov chain consist of  $\boldsymbol{\theta} = (\mathbf{f}, \sigma_\epsilon^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2, \nu)$ .
  - 2: Initialize the algorithm with a configuration  $\boldsymbol{\theta}^{(0)} = (\mathbf{f}^{(0)}, \sigma_\epsilon^{2(0)}, \Sigma_{\mathbf{f}}^{(0)}, \sigma_{\mathbf{f}}^{2(0)}, \nu^{(0)})$ .
  - 3: **procedure** For the  $l$ -th iteration of the algorithm,  $l = 1, \dots, L$
  - 4:   Update  $\boldsymbol{\theta}_{-\nu}$  according to Algorithm 1;
  - 5:   Update  $\nu$  according to Algorithm 2;
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## 2.1 Predictions

Defined the estimation procedure for the parameters  $\boldsymbol{\theta}$ , another interest lies in predicting the value  $f_{n+1} = f(x_{n+1})$  for a new input  $x_{n+1}$ . That is, the interest is in the predictive distribution

$$\pi(y_{n+1}|\mathbf{y}) = \int \pi(y_{n+1}|\mathbf{y}, \boldsymbol{\theta})\pi(\boldsymbol{\theta}|\mathbf{y})d\boldsymbol{\theta}, \quad (10)$$

where  $\pi(\boldsymbol{\theta}|\mathbf{y})$  is the joint posterior distribution for  $\boldsymbol{\theta}$ , given in Equation (3). However, this integrals does not have a known mathematical solution. Due to this, we present in the following an MCMC algorithm for getting an approximation for this integral.

From model (1), the marginal distribution for  $\mathbf{Y}$  is given by a  $n$ -variate normal distribution with mean vector  $\mathbf{0}$  and covariance matrix  $\Sigma_{\mathbf{f}} + \Sigma_\epsilon$ ,  $\mathbf{Y}|\Sigma_{\mathbf{f}}, \sigma_\epsilon^2 \sim \mathcal{N}_n(\mathbf{0}, \Sigma_{\mathbf{f}} + \Sigma_\epsilon)$ ; see Appendix 1 for details. Since  $f_{n+1} \sim \mathcal{N}(m_{n+1}, \sigma_{n+1}^2)$ , the joint distribution for  $(\mathbf{Y}, f_{n+1})$  is

$$\begin{bmatrix} \mathbf{Y} \\ f_{n+1} \end{bmatrix} | \mathbf{f}, \mathbf{x}, x_{n+1}, \Upsilon \sim \mathcal{N}_{n+1} \left( \begin{bmatrix} \mathbf{f} \\ m(x_{n+1}) \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathbf{f}} + \Sigma_\epsilon & \mathbb{C}^T \\ \mathbb{C} & \sigma_{n+1}^2 \end{bmatrix} \right),$$

where  $\Upsilon = (\Sigma_{\mathbf{f}}, \sigma_\epsilon^2, \sigma_{n+1}^2, \mathbb{C})$  and  $\mathbb{C} = (\sigma(1, n+1), \dots, \sigma(n, n+1))$  is a row vector,  $1 \times n$ , composed by the covariance among  $f_t$  and  $f_{n+1}$ ,  $\sigma(t, n+1) = cov(f_t, f_{n+1})$ , for  $t = 1, \dots, n$ .

By using the properties of the multivariate normal distribution, the conditional posterior distribution for  $f_{n+1}$  is given by

$$f_{n+1} | \mathbf{y}, \mathbf{f}, \mathbf{x}, x_{n+1}, \Upsilon \sim \mathcal{N} \left( m(t_{n+1}) + \mathbb{C} (\Sigma_{\mathbf{f}} + \Sigma_\epsilon)^{-1} (\mathbf{y} - \mathbf{f}), \sigma_{n+1}^2 - \mathbb{C} (\Sigma_{\mathbf{f}} + \Sigma_\epsilon)^{-1} \mathbb{C}^T \right). \quad (11)$$

At this point, as we know the value of the 7-day moving average in the  $n$ -th day, we fix up  $m(t_{n+1}) = y_n$ . In addition, we set up each  $\sigma(t, n+1) = k(t, n+1)$  in order to get these values according to Equation (2). Then, a sample from conditional posterior

distribution in (11) can be generated according to the implementation of the Algorithm 4.

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**Algorithm 4** Prediction
 

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- 1: Let the state of the Markov chain consist of  $\boldsymbol{\theta} = (\mathbf{f}, \sigma_\epsilon^2, \Sigma_{\mathbf{f}}, \sigma_{\mathbf{f}}^2, \nu)$  and  $f_{n+1}$ .
  - 2: Initialize the algorithm with a configuration  $\boldsymbol{\theta}^{(0)} = (\mathbf{f}^{(0)}, \sigma_\epsilon^{2(0)}, \Sigma_{\mathbf{f}}^{(0)}, \sigma_{\mathbf{f}}^{2(0)}, \nu^{(0)})$  and  $f_{n+1}^{(0)}$ .
  - 3: **procedure** For the  $l$ -th iteration of the algorithm,  $l = 1, \dots, L$
  - 4:   Update  $\boldsymbol{\theta}$  according to Algorithm 1;
  - 5:   Update  $\nu$  according to Algorithm 2;
  - 6:   Conditional on  $\boldsymbol{\theta}^{(l)} = \boldsymbol{\theta}^{(l)} = (\mathbf{f}^{(l)}, \sigma_\epsilon^{2(l)}, \Sigma_{\mathbf{f}}^{(l)}, \sigma_{\mathbf{f}}^{2(l)}, \nu^{(l)})$ , generate  $f_{n+1}^{(l)}$  from conditional posterior distribution in (11).
- 

After running the algorithm for the same  $L$  iterations, burn in  $B$  and jump  $J$  of the algorithm (3), an approximation for the integral in (10) is given by

$$\tilde{\pi}(f_{n+1}|\mathbf{y}) = \sum_{l=1}^L f_{n+1}^{(M(l))},$$

where  $M(l)$  is the  $(B + 1 + l \cdot J)$ -th iteration of the algorithm.

The predictions for the next  $n + j$  days, for  $j = 2, \dots, r$ , is obtained in a similar way by generating a sample from  $f_{n+j}$  according to Algorithm 4, where  $f_{n+j}$  is given by Equation (11), setting up  $f_{n+j} = \hat{f}_{n+j-1}$  and  $\mathbb{C} = (k(1, n + j), \dots, k(n, n + j))$ , for  $j = 2, \dots, r$ .

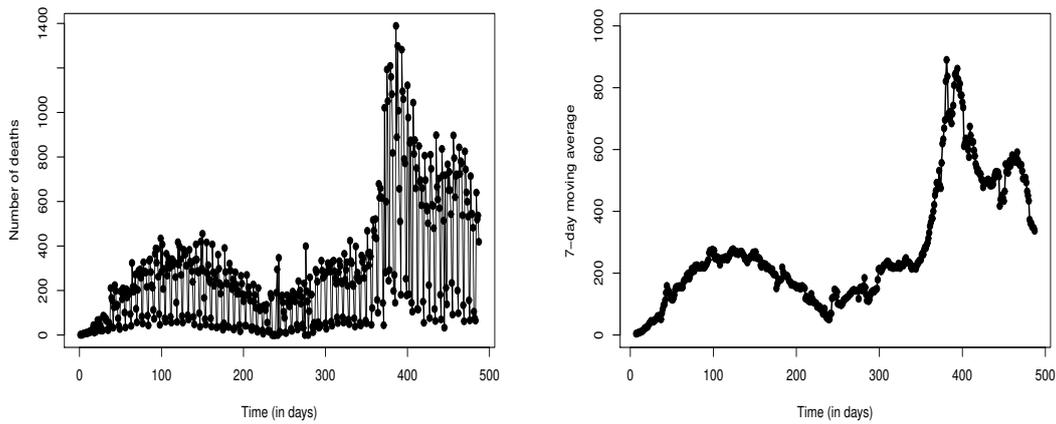
In many cases, the interest does lies in predicting the value  $f_{n+1} = f(x_{n+1})$  for a new input  $x_{n+1}$  conditional on the last  $\mathbf{y}_s$  recorded values and not on all past values  $\mathbf{y}$ . For example, one can have interest in the value  $y_{n+1}$  given the recorded 7-day moving average in the last  $s = 30$  days,  $\mathbf{y}_{30} = (y_{n-29}, \dots, y_n)$ . This prediction is also made using Algorithm 4, just changing  $\mathbf{y}$  by  $\mathbf{y}_s$  and adapting the parameters  $\boldsymbol{\theta}$  for dimension of  $\mathbf{y}_s$ . The predictions for the next  $n + j$  days is done as described in the paragraph above, but with some obvious differences. The advantage of this kind of prediction is the computation time that is very smaller than the prediction procedure conditional on  $\mathbf{y}$ .

### 3. Application

In this section, we apply the proposed method to a real dataset. The dataset refers to the 7-day moving average for the number of deaths recorded in the state of São Paulo, Brazil. As an illustration of this dataset, we present in Figure 1 the number of deaths recorded in the period from 03/17/2020 (first case) to 07/16/2021 and the 7-day moving average recorded in the period from 03/23/2020 ( $t = 0$ ) to 04/27/2021 ( $n = 480$ ).

In order to estimate parameters of interest and make the predictions, we apply Algorithms 3 and 4 with  $L = 55,000$  iterations,  $B = 5,000$  iterations and  $J = 10$ . Thus, we got a sample of size 1,000 to make inferences. Using these values, we ran our first analysis on the 100th day after the recording of the first 7-day moving average. Figure 2 shows the observed value (symbols  $\bullet$ ) and the credibility band of 95% (blue area) determined by the proposed method. This Figure also shows the prediction band of 95% (black area) and the recorded values for the next ten days. The mean square error (MSE) of the predicted values in relation to the recorded values was 0.0056. As one can note,

the prediction band contains all recorded values. Both results show a very satisfactory performance of the proposed method.



(a) Number of deaths. (b) 7-day moving average.

Figure 1. Number of deaths by day and 7-day moving average values.

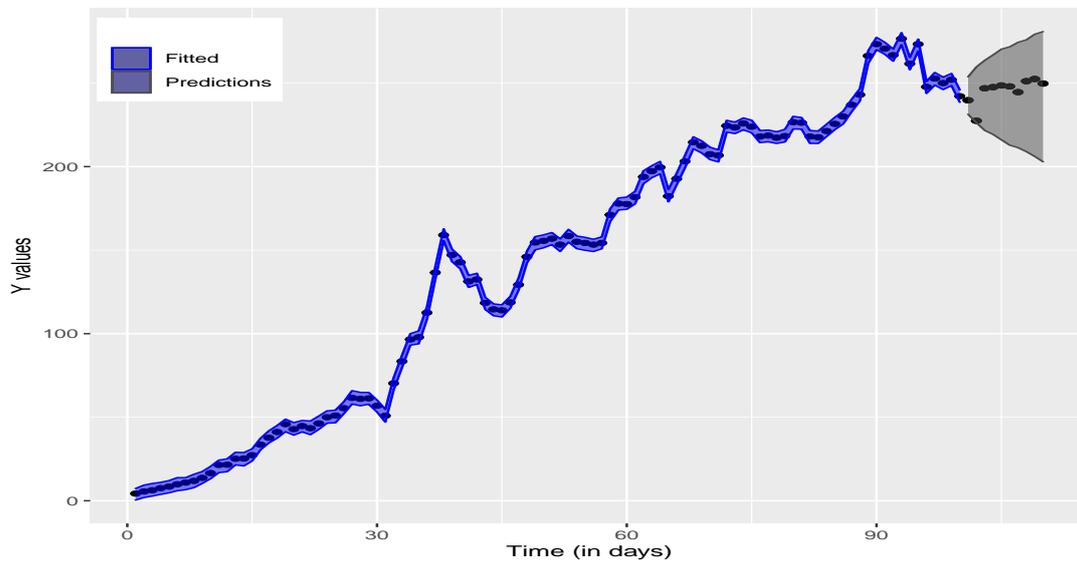


Figure 2. Confidence and prediction bands (95%).

The estimated value for the  $\sigma_\epsilon^2$  is  $\hat{\sigma}_\epsilon^2 = 2.4427$ , with a credibility interval (95%) give by (2.0154, 2.9632). Figure 3 show the ergodic mean (ErM) and the estimated autocorrelation function (ACF) from the sampled values for  $\sigma_\epsilon^2$ . As one can note, there is no reason to doubt the convergence of the sampled values, since the ErM values present satisfactory stabilization and there is no significant ACF.

We also verify the convergence of the sampled values for  $\mathbf{f}$ . The results are similar to the presented by the sampled values for  $\sigma_\epsilon^2$ . As an illustration, we show in Figure 4 the ErM and the ACF for the sample value for  $f_{22}$ . The  $f_{22}$  was chosen at random among the  $f_t \in \mathbf{f}$ , for  $t = 1, \dots, n$ . The estimate for  $f_{22}$  is  $\hat{f}_{22} = 43.7603$ ; and the credibility interval (95%) is (40.2492, 49.2031). The recorded value was 43.43. That is an absolute percentage error of 0.7604%. Analogously to the sampled value for  $\sigma_\epsilon^2$ , there is no reason to doubt the convergence of the sampled values for  $f_{22}$ .

The estimate for parameter  $\sigma_f^2$  is  $\hat{\sigma}_f^2 = 3.8234$  with credibility interval (2.0459, 3.9189). The estimate for parameter  $\nu$  is  $\hat{\nu} = 0.2637$  with credibility interval (0.0153, 0.4982) and acceptance rate of 26.78%. The convergence checking for both parameters is similar to the presented for  $\sigma_\epsilon^2$  and  $f_{22}$ , *i.e.*, the ergodic mean presents satisfactory stabilization and there is no significant ACF.

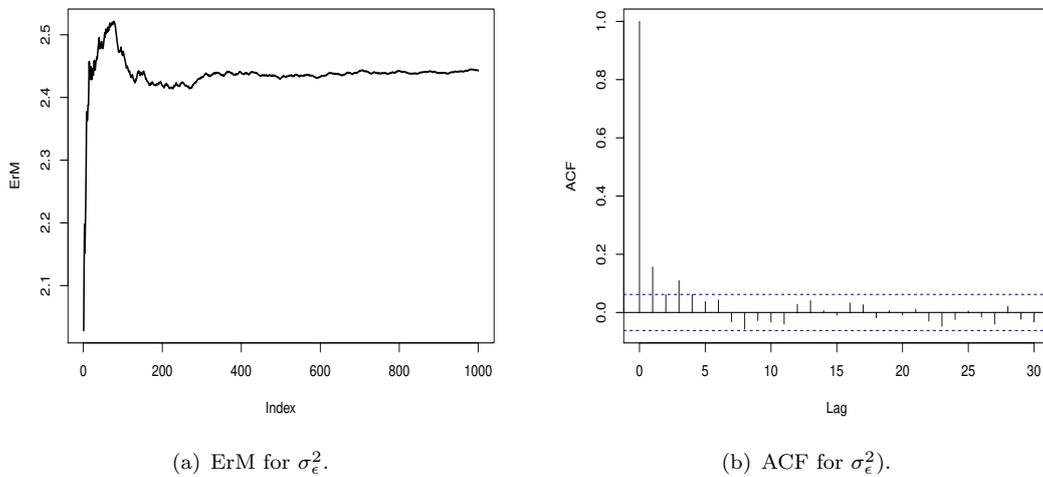


Figure 3. ErM and ACF from the sampled values for  $\sigma_\epsilon^2$ .

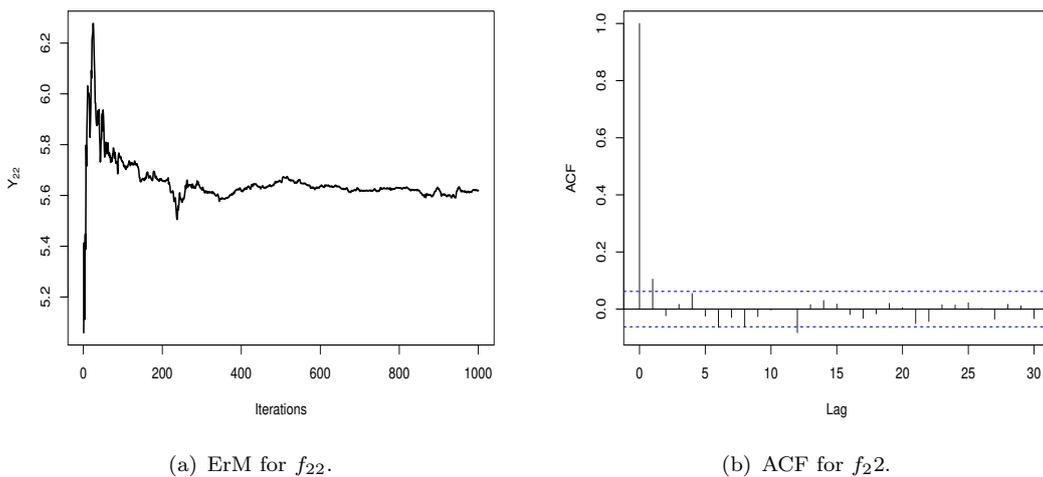


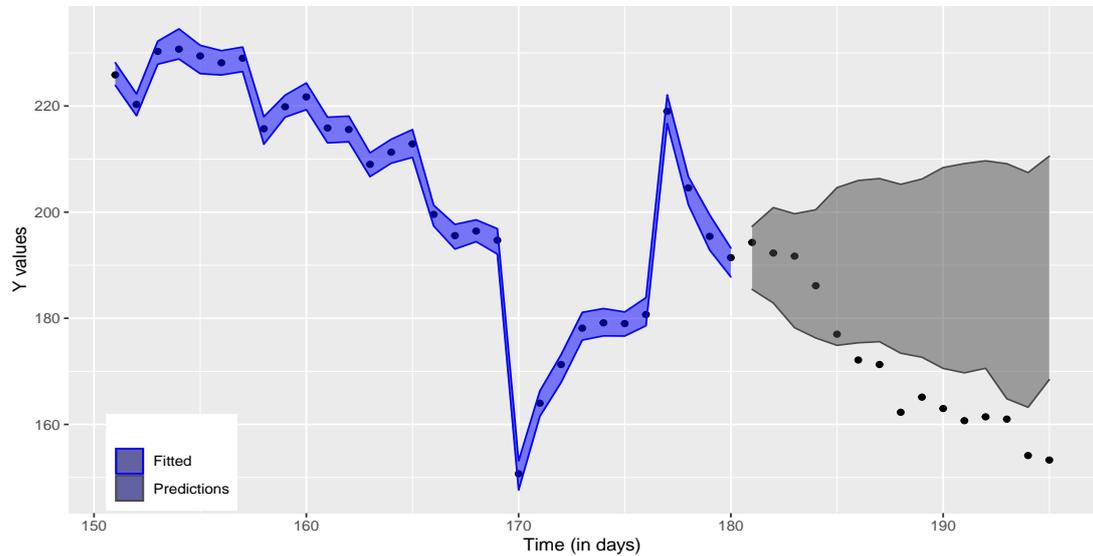
Figure 4. ErM and ACF from the sampled values for  $f_{22}$ .

### 3.1 Predictions based in the last 30 days

Consider now the interest in predict the evolution of the 7-day moving average values for the next 15 days after the  $n$ -th day conditional on the recorded values in the last thirty days. That is, the interest is to predict the values  $f_{n+j}$  for  $j = 1, \dots, 15$  conditional on  $\mathbf{y}_{30} = (y_{n-30+1}, \dots, y_n)$ . In the following, we present the results from three analyses ran at 180-th, 280-th and 465-th days.

In order to obtain these predictions, we apply Algorithm 4 just changing  $\mathbf{y}$  by  $\mathbf{y}_{30}$  and adapting the dimension of the parameters  $\boldsymbol{\theta}$ . We ran this algorithm for the same values for  $L$ , burn-in and jump  $J$  used in section 3. Figure 5 show the results of the analysis

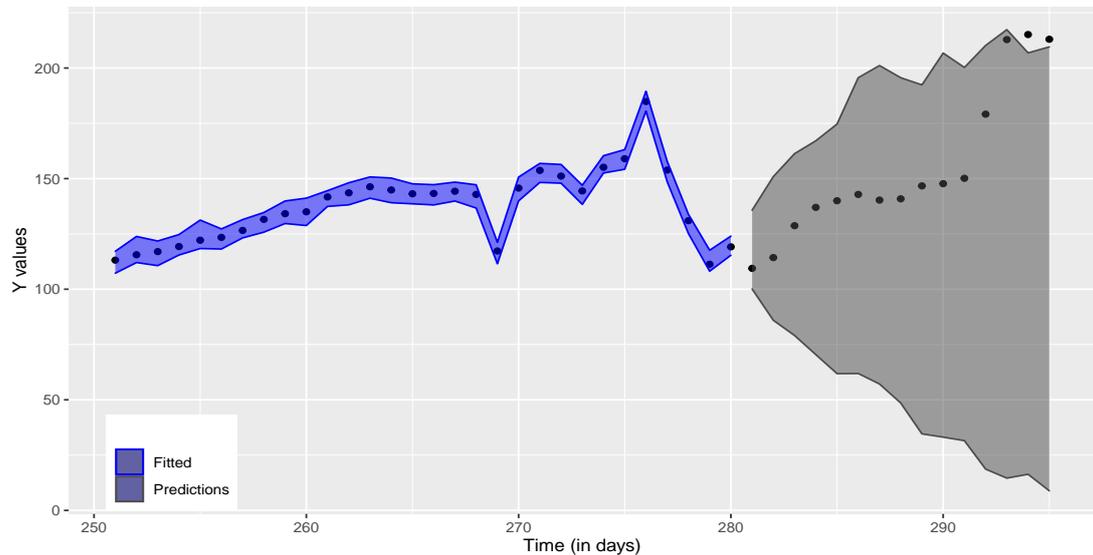
carried out on the 180-th day. The MSE of the fitted model is 0.0123. As one can note, the five 7-day average values recorded after the 180-th day is inside the prediction band; but the values recorded after the 185-th day are outside the band. This a good news because it indicates that the reduction in the values of the 7-day moving average was greater than expected. In this period of 15 days, the moving average reduced from 191.43 (180th day) to 153.29 (195-th day). A reduction of 19.42%



(a) 2-nd analysis.

Figure 5. Recorded values and confidence band determined by the proposed method, 180-th day.

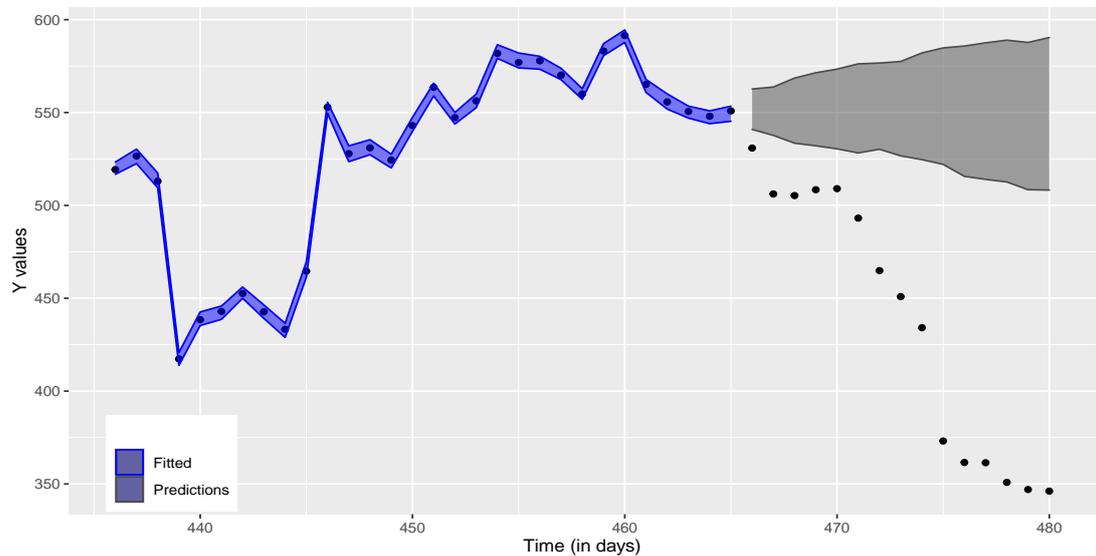
Figure 6 shows the results of the analysis ran on the 280-th day. The MSE of the fitted model is 0.0471. As one can note, only the two last recorded values are outside the prediction band. In this period of 15 days, the 7-day moving average value increased 44.05%, going from 119.14 (280th day) to 213 (295th day).



(a) 2-nd analysis.

Figure 6. Recorded values and confidence band determined by the proposed method.

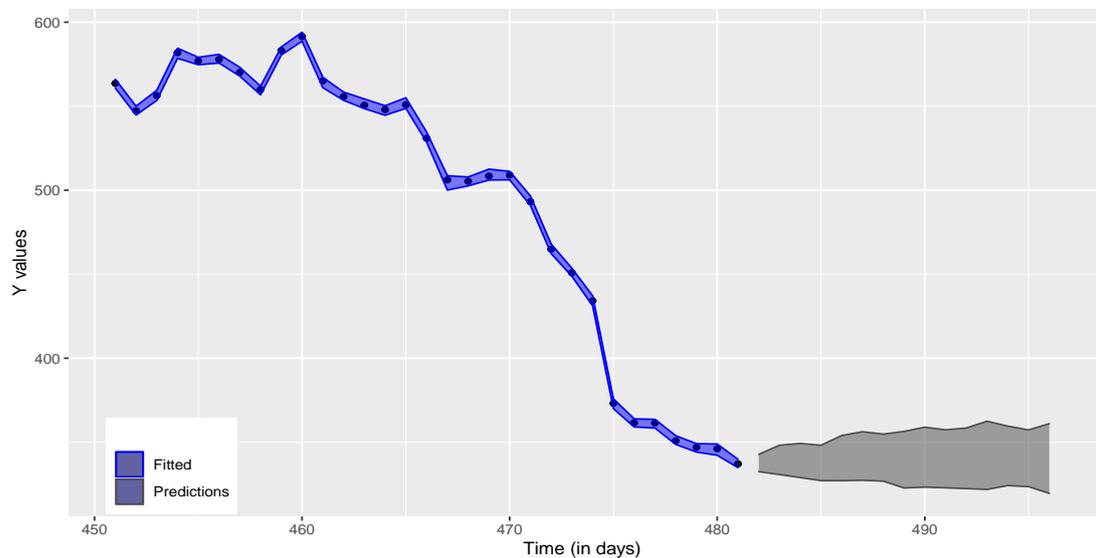
Figure 7 shows the results of the analysis ran on the 465-th day. The MSE of the fitted model is 0.0437. Note that, the credibility band determined by the proposed method base on the recorded values in the last thirty days indicates a stabilization of the moving average, but the recorded values were all below this region. This is a piece of very good news because it shows that a greater reduction than expected happened. The reduction was of 49.14%, going from 550.86 (365-th day) to 340.14 (480-th day).



(a) 2-nd analysis.

Figure 7. Recorded values and confidence band determined by the proposed method, 280-th day.

Figure 8 shows the recorded values in the last 30 days and the credibility band determined by the proposed method for the next 15 days. As one can note, this graphic shows that is not expected a great increase in the values of the 7-day moving average in the next 15 days.



(a) 2-nd analysis.

Figure 8. Recorded values and confidence band determined by the proposed method, 465-th day.

#### 4. Final Remarks

This article presented a hierarchical Bayesian methodology for modeling the evolution of the 7-day moving average for the number of deaths due to COVID-19. We opt to model the 7-day moving average in opposite to the cumulative number of deaths, as is usual, due to moving average smooth the possible discrepancies that may have in the statistical reports of the number of deaths divulged.

Contrary to usual approaches, which are based on the adoption of a set of parametric models followed by a comparison using some model selection criteria, such as, AIC and BIC, the proposed approach assumes that the non-linear mathematical function is an unknown random quantity that needs to be estimated from the observed data. Then, we adopt a Bayesian nonlinear regression approach with a Gaussian process.

The adoption of the Gaussian process prior means that we are defining a probability distribution over functions and the inference is taken directly in the space of the functions. This approach is considered nonparametric since we are using a prior distribution on the space of functions that corresponds to infinite-dimensional parameter space. Since making inferences about the infinite number of parameters is impractical, the assumption of the Gaussian process has the advantage of that conditional on a dataset a finite number of parameters (function values) can be explicitly represented by a multivariate normal distribution, making the estimation procedure simple to be computationally implemented.

In addition, since the Gaussian process is heavily influenced by the choice of the covariance function  $\Sigma_f$ , we assume one more hierarchical level by putting a prior distribution over  $\Sigma_f$  and on its hyperparameters. The inference on the parameters of interest is carried out using a Metropolis-within-Gibbs algorithm. We also present an MCMC algorithm to make predictions.

The proposed method is illustrated in a case study, in which, we model the 7-day moving average values recorded in the state of São Paulo, Brazil, in the period from 03/23/2020 to 04/27/2021. In this period, we run the estimation procedure four times in order to verify the performance of the method. The results have shown a very good performance of the method. The MSE values were all smaller than 0.05 and there is no reason to doubt of the convergence of the sampled values by the MCMC algorithm.

From a statistical data analysis point of view, the proposed method is very interesting because it does not need the assumption of a parametric model and the inference is done by an MCMC algorithm that can be easily implemented in free softwares, such that, R software. The computational codes were implemented in the R language and can be obtained by e-mail to the authors.

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