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Life Distributions: A Brief Discussion

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This article presents a brief description of some of the characteristics of life distributions that arise in survival analysis and reliability theory. Alternative definitions of a distribution are discussed and related to a variety of stochastic orders: hazard rate order, likelihood ratio order, convex order, Lorenz order. Nonparametric families, particularly log-concave densities, completely monotone distributions, increasing hazard rate families, new-better-than-used families, and bathtub hazard rate families are analyzed. A taxonomy for semiparametric families is presented and the effect of introducing parameters on various stochastic orders is shown. Finally, the introduction of covariate models in these families is developed.

Keywords Hazard rates; Log-concave densities; Nonparametric families; Reliability theory; Stochastic orders; Survival analysis.

Mathematics Subject Classification 62N05; 60E02.

1. Introduction

Sir Ronald Fisher as early as 1925 stated a motivation for selecting models: "From a limited experience, for example, of individuals of a species, or of the weather of a locality, we may obtain some idea of the infinite hypothetical population from which our sample is drawn, and so of the probable nature of future samples to which our conclusions are to be applied" (Fisher, 1925).

The fitting of distributions goes back many years with the construction of general families such as the Pearson family, Chebyshev–Hermite polynomials, Gram–Charlier series, Edgeworth series, and many more. The more recent research on splines and density estimation falls into this category.

However, an alternative to fitting distributions was enumerated by Kingman (1978): "Although it is often possible to justify the use of a distribution empirically, simply because it appears to fit the data, it is more satisfactory if the structure of the distribution reflects plausible features of the underlying mechanism." The present work endeavors to follow this theme by clarifying characteristics of distributions that one might consider *before* fitting a distribution. The development of the Gompertz distribution from actuarial tables provides an example in which the hazard rate was used. The search for comparing inequality of wealth was used to study the Lorenz order. Of course, the Poisson distribution is a standard distribution that arose from models.

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2. Alternative Definitions of Distributions

There is no unique description of a distribution. Some forms provide more intuition about certain characteristics than other forms. It is important to recognize that each description provides a different view and understanding of the data. The most common definitions are the following.

1. The cumulative distribution function (cdf) F defined on the interval $(-\infty, \infty)$ by
$$F(x) = P\{X \le x\}.$$
2. The survival function $\bar{F}(x) = 1 - F(x)$. Some distributions such as the exponential or Weibull distributions have simple expressions for the survival function. For lifetime data the survival function is more easily interpreted.
3. If f is a nonnegative function for which
$$F(x) = \int_{-\infty}^{x} f(z)\, dz$$
for all real x, then f is called the density of X (or F). Fitting densities to data has a long history. Traditionally families of curves or series have been developed to aid in fitting densities. Of course, current numerical procedures, such as splines, are generally available for fitting densities; see, e.g., Scott (1992), Silverman (1986), Klemelä (2009). However, we note that the cdf in item 1 above is unsuited to determine whether a density is unimodal, and similarly the Kolmogorov–Smirnov statistic is suited for the cdf and not for the density.
4. The hazard function $R(x) = -\log \bar{F}(x)$ defined on $(-\infty, \infty)$. Hazard functions are useful mathematical tools in part because they are linear for exponential distributions.
5. If F is an absolutely continuous distribution function with density f, then the function r defined on $(-\infty, \infty)$ by
$$r(x) = \begin{cases} f(x)/\bar{F}(x) & \text{if } \bar{F}(x) > 0, \\ \infty & \text{if } \bar{F}(x) = 0, \end{cases}$$
is called the hazard rate. The hazard rate is a fundamental concept in survival analysis. The determination of an increasing or decreasing hazard rate is not readily observed from a graph of the cdf or the density. The connection between the hazard rate and hazard function is given by
$$R(x) = \int_{0}^{x} r(t)\, dt.$$
6. Let F be a distribution function such that $F(0) = 0$. The residual life distribution F_t of F at t is defined for all $t \ge 0$ such that $\bar{F}(t) > 0$ by
$$\begin{aligned} \bar{F}_t(x) &= \bar{F}(x+t)\big/\bar{F}(t) \\ &= P\{X > x + t \mid X > t\}. \end{aligned}$$
The residual life distribution arises naturally as a characterization of the exponential distribution. This is also true of the hazard rate. This formulation is fundamental in the construction of actuarial tables.

7. If F has a finite mean and $F(x) = 0$ for $x < 0$, then the mean residual life (mrl) is given by

$$m(t) = \begin{cases} \int_0^\infty \bar{F}_t(x)\,dx & \text{for } t \text{ such that } \bar{F}(t) > 0, \\ 0 & \text{for } \bar{F}(t) = 0. \end{cases}$$

The mrl is related to the survival function by

$$\bar{F}(x) = \frac{\mu}{m(x)} \exp\left\{-\int_0^x \frac{dz}{m(z)}\right\},$$

where μ is the mean of the distribution.

8. The inverse distribution

$$F^{-1}(p) = \sup\{z : F(z) \le p\}, \quad F(z) = \inf\{p : F^{-1}(p) \ge z\}.$$

Inverse distributions are essentially the same as quantiles and play an important role in statistical inference. See Parzen (2004) for a survey of quantile functions.

9. The function $\omega^+(x)$ defined for x such that $F(x) > 0$ by

$$\omega^+(x) = \frac{\bar{F}(x)}{F(x)}$$

is called the odds ratio of survival.

10. The function $\omega^-(x)$ defined for x such that $\bar{F}(x) > 0$ by

$$\omega^-(x) = \frac{F(x)}{\bar{F}(x)}$$

is called the odds ratio of failure. The odds ratio of failure of the residual life distribution F_t has the property that it is increasing (decreasing) if and only if the hazard rate of the distribution F is increasing (decreasing).

Each of these alternative definitions provides different insights into the data. Although these alternative definitions apply to the univariate case, only a few generalize to the bivariate case, which may explain the use of the more common cdf or density function.

3. Ordering of Distributions

Characteristics of distributions such as location, dispersion, skewness, kurtosis, long tailed, unimodality, etc., have long been used for descriptive purposes. Alternatively distributional characteristics can be described in terms of a variety of the ordering of random variables. These orders provide insight into magnitude, shape, or hazard rates. For a more complete discussion of orders, see Shaked and Shanthikumar (2007), Szekli (1995), or Müller and Stoyan (2002).

1. Stochastic order $X \underset{\text{st}}{\le} Y$ or $F \underset{\text{st}}{\le} G$:

$$P\{X \ge z\} \le P\{Y \ge z\}.$$

This order is a tail order introduced by Mann and Whitney (1947). It remains as the most well-known ordering of distributions.

2. Hazard rate order $X \underset{\text{hr}}{\leq} Y$ or $F \underset{\text{hr}}{\leq} G$:

$$r_X(z) \geq r_Y(z).$$

3. Likelihood ratio order $X \underset{\text{lr}}{\leq} Y$ or $F \underset{\text{lr}}{\leq} G$:

$$\frac{f(u)}{f(v)} \geq \frac{g(u)}{g(v)} \qquad \text{for all } u \leq v,$$

   where f and g are the respective densities of F and G. Although the likelihood ratio order does not have an intuitive interpretation, it is a useful order because it is easier to prove than stochastic order, and implies stochastic order.
4. Convex order $X \underset{\text{cx}}{\leq} Y$ or $F \underset{\text{cx}}{\leq} G$:

$$E\phi(X) \leq E\phi(Y)$$

   for all convex functions ϕ. This order has proved useful because the class of convex functions is so rich.
5. Lorenz order $X \underset{\text{L}}{\leq} Y$ or $F \underset{\text{L}}{\leq} G$:

$$\frac{X}{EX} \underset{\text{cx}}{\leq} \frac{Y}{EY}.$$

   This order was introduced by Lorenz (1905) and is a central ordering in economics, especially for comparing the wealth of two communities. For applications of the Lorenz order and its relation to the Gini index, see Yitzhaki and Schechtman (2013).

4. Mixtures and Stochastic Order

Because so many distributions arise as mixtures from several populations, it is comforting to know that stochastic order is preserved under mixture. Mixtures also play a central role in Bayesian statistics.

If $X \underset{\text{st}}{\leq} Y$ and $U \underset{\text{st}}{\leq} V$, and we mix X with U and Y with V,

$$W = IX + \bar{I}U,$$
$$Z = IY + \bar{I}V,$$

where I is a random variable independent of X, Y, U, and V such that $P\{I = 1\} = 1 - P\{I = 0\}$, then

$$W \underset{\text{st}}{\leq} Z.$$

From the point of view of applications, the mixing random variable I will generally represent a discrete covariate.

5. Nonparametric Families

5.1. *Logconcave Densities*

One of the most important classes of nonparametric families is that of logconcave densities,

$$f(x) = \exp\{-\phi(x)\},$$

where ϕ is convex. Examples are the normal, gamma, Weibull densities. They also play a role in economic theory; see An (1998).

Logconcave densities have the following properties:

(i) The density is unimodal.
(ii) The convolution of two logconcave densities is logconcave.
(iii) The cdf and survival functions of logconcave distributions are logconcave.
(iv) The distribution has an increasing hazard rate.

5.2. *Completely Monotone Distribution*

This class was defined by Feller (1971, p. 431):

$$\bar{F}(x) = \int_0^\infty e^{-\lambda x}\, dH(\lambda),$$

where H is a distribution function. Thus, $\bar{F}$ is a mixture of exponential survival functions. The key properties of this family are that if $\bar{F}$ is completely monotone, then F is infinitely divisible, and the density is logconvex on $[0, \infty]$.

5.3. *Increasing Hazard Rate (IHR) Families*

As noted, the hazard rate is one of the most important characteristics of distributions used in survival or reliability analysis.

(i) A distribution F has IHR if and only if $-\log F$ is convex.
(ii) A distribution F has IHR if and only if the residual life distribution

$$\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t)$$

is decreasing.
(iii) An important fact is that if F and G have IHR, then the convolution $F * G$ has an IHR.
(iv) IHR families have finite moments of all positive orders.
(v) If F has IHR then $\bar{F}$ is a Pólya frequency function of order 2.

A somewhat disturbing feature is that mixtures of distributions with IHR can have decreasing hazard rates (DHRs). This is important when modeling with covariates that can create mixtures. Of singular importance is that a mixture of exponential distributions (each of which has a constant hazard rate) has a DHR.

5.4. *Increasing Hazard Rate Average (IHRA) Families*

Suppose there is a system of components, then the repair of a failed component should not cause the system to fail. This led to the notion of a coherent system. A binary function ϕ of

n binary variables is called a *coherent system* if $\phi(0, \ldots, 0) = 0$, $\phi(1, \ldots, 1) = 1$, and ϕ is increasing in each of its arguments.

A rather disturbing fact was uncovered, namely, that the class of IHR distributions is not closed under the formation of coherent systems. This led to the introduction of the class of IHRA families, which is closed under the formation of coherent systems.

A condition weaker than IHR is an average: a distribution F satisfying $F(0) = 0$ is said to have an IHRA if the hazard function $R = -\log F$ satisfies $R(t)/t$ increasing in $t > 0$. The terminology arises from its relation to the hazard rate:

$$\frac{R(t)}{t} = \frac{1}{t}\int_0^t r(z)\,dz,$$

IHRA families are closed under convolutions, but not under mixtures.

5.5. *New Better than Used (NBU) Families*

The idea of NBU families is intuitive in that it compares the residual life distribution with the distribution,

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} \leq \bar{F}(x),$$

or in terms of probabilities,

$$\frac{P\{X > x+t\}}{P\{X > t\}} \leq P\{X \geq x\}.$$

Two important properties of NBU families are that the convolution of two NBU families is NBU, and that if the components of a coherent system are NBU, then the system life distribution is NBU.

Although the construction of NBU families arose from an engineering context, as in the case of airplanes, the concept now can be applied to organ transplants in medicine. The following is an abbreviated summary of implications:

$$\text{logconcave density} \Longrightarrow \text{IHR} \Longrightarrow \text{IHRA} \Longrightarrow \text{NBU}$$

6. Bathtub-shaped Hazard Rates

An examination of some of the common distributions in survival analysis and reliability such as the exponential, Weibull, gamma, lognormal, Gompertz distributions shows that they have either increasing, decreasing, or constant hazard rates. Yet we know that natural phenomena may have bathtub-shaped hazard rates that traverse a decreasing phase, a constant phase, and an increasing phase. The construction of distributions with a bathtub hazard rate was illusive because they do not arise from a single distribution.

The following provides two constructions of distributions with bathtub-shaped hazard rates. However, knowledge of the data is essential in deciding which construction provides a reasonable model.

1. If

$$X = X_0 \qquad \text{with probability } \pi$$

$$= \min(X_0, X_1) \qquad \text{with probability } \bar{\pi} = 1 - \pi,$$

then the hazard rate for the distribution of X can have a bathtub hazard rate. An example is given by two Weibull distributions

$$\bar{F}_0(x) = \exp\{-x^{\alpha}\}, \qquad \bar{F}_1(x) = \exp\{-x^{\beta}\}.$$

The hazard rate will be bathtub for $\alpha \geq 2$ and $\beta \leq 1$.

2. A second construction is based on

$$X = \min(U, V, W),$$

where U has a DHR, V has an IHR, and W has a constant hazard rate. Because the hazard rate of X is a sum of the hazard rates

$$r(x) = r_U(x) + r_V(x) + r_W(x),$$

one increasing, one decreasing, and one constant, it is clear that various shapes can be generated.

7. Taxonomy of Semiparametric Families

A semiparametric family is constructed by the insertion of a parameter into a family $F(x)$ or $\bar{F}(x)$. There are many ways to add a parameter to a distribution. The following lists a number of such parameters, each with a descriptor.

1. Location $\bar{F}(\mu) = \bar{F}(x - \mu)$.
2. Scale $\bar{F}(x|\lambda) = \bar{F}(\lambda x), \quad \lambda > 0$.
3. Power $\bar{F}(x|\alpha) = \bar{F}(x^{\alpha}), \quad \alpha > 0$.
4. Frailty or proportional hazards $\bar{F}(x|\xi) = \left[\bar{F}(x)\right]^{\xi}, \quad \xi > 0$.
5. Resilience $F(x|\eta) = [F(x)]^{\eta}, \quad \eta > 0$.
6. Tilt $\bar{F}(x|\gamma) = \gamma\bar{F}(x)\big/\left[1 - \bar{\gamma}\bar{F}(x)\right]$.
7. Moment or shape $f(x|\beta) = x^{\beta} f(x)/\mu_{\beta}$, where μ_{β} is a normalizing constant.
8. Age $\bar{F}(x|\tau) = \bar{F}(x + \tau)\big/\bar{F}(\tau), \quad \tau > 0$.

Parametric families can be constructed by introducing parameters in some sequence. For example, if $\bar{F}(x) = \exp\{-x\}$, then the introduction of a scale parameter yields $\bar{F}(x) = \exp\{-\lambda x\}$, then a power parameter $\bar{F}(x) = \exp\{-\lambda x^{\alpha}\}$, followed by a moment or shape parameter to yield the density

$$f(x) = cx^{\alpha\beta} \exp\{-\lambda x^{\alpha}\},$$

where $c = c(\alpha, \beta, \lambda)$ is a normalizing constant. This density is that of a generalized gamma distribution.

In another example suppose one starts with a Pareto family $\bar{F}(x) = 1/(1 + x)$, then introduce a scale parameter $\bar{F}(x) = 1/(1+\lambda x)$, then a frailty parameter $\bar{F}(x) = 1/(1+\lambda x)^{\xi}$, and finally a moment and power parameter to yield the density

$$f(x) = c\frac{(\lambda x)^{\alpha\theta-1}}{[1 + (\lambda x)^{\alpha}]^{\xi}},$$

where $c = c(\lambda, \alpha, \xi, \theta)$ is a normalizing constant. This density is called a generalized F-distribution.

The introduction of a succession of different parameters may lead to the same family in which case the order does not matter. This is the case with power and scale parameters:

1. $\bar{F}(x) \to \bar{F}(\lambda x) \to \bar{F}(\lambda x^{\alpha})$,
2. $\bar{F}(x) \to \bar{F}(x^{\alpha}) \to \bar{F}((\lambda x)^{\alpha})$.

However, with other pairs of parameters the succession may lead to different families. This is the case with frailty and resilience families.

1. $\bar{F}(x) \to \left[\bar{F}(x)\right]^{\xi} \to 1 - (1 - \left[\bar{F}(x)\right]^{\xi})^{\eta}$,
2. $\bar{F}(x) \to 1 - [F(x)]^{\eta} \to \{1 - [F(x)]^{\eta}\}^{\xi}$.

8. Coincidences

The question posed here is what distributions are contained in the coincidence of two families? For example, the coincidence of the scale family $\bar{F}(\lambda x)$ and frailty family $[\bar{F}(x)]^{\xi}$ is the solution of the functional equation

$$\bar{F}(\lambda x) = \left[\bar{F}(x)\right]^{\xi(\lambda)}.$$

The solution is the Weibull distribution

$$\bar{F}(x) = \exp\left[-(\lambda x)^{\alpha}\right], \qquad x > 0,$$

or the extreme value distribution

$$\bar{F}(x) = \exp\left[-(\lambda x)^{-\alpha}\right], \qquad x < 0.$$

The lognormal distribution arises from the coincidence of the scale and moment families and is the solution of the functional equation

$$\lambda f(\lambda x) = \frac{x^{\beta} f(x)}{c(\beta)},$$

where $c(\beta)$ is a normalizing constant.

The Pareto distribution arises from the coincidence of the scale and tilt families and is the solution of the functional equation

$$\bar{F}(\lambda x) = \frac{\gamma(\lambda)\bar{F}(x)}{1 - \bar{\gamma}(\lambda)\bar{F}(x)}.$$

9. Effect of the Introduction of a Parameter on Stochastic Orders

Suppose $F \le G$ in one of the many stochastic orders, and a parameter is introduced. Will the new parametric distributions still be ordered in the same way? The answer is somewhat surprising.

If $F \le G$ in stochastic order, hazard rate order, likelihood ratio order, convex order, Lorenz order, and a scale parameter is introduced, then for the scale parameter family $F(\cdot|\lambda) \le G(\cdot|\lambda)$ in the same order.

If $F \le G$ in stochastic or hazard rate order and a power parameter is introduced, then for the power parameter family $F(\cdot|\alpha) \le G(\cdot|\alpha)$ in the same order, and for the frailty or proportional hazards family $\bar{F}(\cdot|\xi) \le \bar{G}(\cdot|\xi)$ in the same order.

10. Proportional Hazards and Proportional Odds Families

The proportional hazards family

$$\bar{F}(x|\xi) = \left[\bar{F}(x)\right]^{\xi}$$

has gained popularity because the hazard rates are proportional:

$$r(x|\xi) = \xi r(x).$$

The parameter ξ is often modeled as a function of covariates.

The family

$$\bar{F}(x|\gamma) = \frac{\gamma \bar{F}(x)}{1 - \bar{\gamma}\bar{F}(x)}, \qquad \bar{\gamma} = 1 - \gamma,$$

is a proportional odds family, that is,

$$\frac{F(x|\gamma)}{\bar{F}(x|\gamma)} = \gamma \frac{F(x)}{\bar{F}(x)}.$$

This family is somewhat new and has not been used much in applications. It would be interesting to compare both the proportional hazards and proportional odds families in some applications.

11. Covariate Models

Most applications involve covariates, and one way to deal with covariates is to model the parameter as a function of the covariates. One of the early examples by Zelen (1959) was based on the exponential distribution

$$\bar{F}(x|\theta) = \exp\{-x/\theta\}$$

in which the scale parameter θ is a function of covariates $z_1, \ldots, z_K$,

$$\theta = \theta(z) = \exp\{\beta_1 z_1 + \ldots + \beta_K z_K\}.$$

The proportional hazards model

$$\bar{F}(x|\xi) = \left[\bar{F}(x)\right]^{\xi}$$

in which the parameter θ is a function of covariates $z_1, \ldots, z_K$,

$$\xi = \xi(z) = \beta_1 z_1 + \ldots + \beta_K z_K$$

has been used in many applications.

Bailey and Homer (1977) used a covariate model for the hazard rate,

$$r(x) = \alpha e^{-\lambda x} + \delta,$$

in which the three parameters $\alpha = \alpha(z_1, \ldots, z_K)$, $\lambda = \lambda(z_1, \ldots, z_K)$, and $\delta = \delta(z_1, \ldots, z_K)$ are functions of covariates.

12. Discussion

This article contains some of the main ideas that were the basis of the book *Life Distributions. Structure of Nonparametric, Semiparametric and Parametric Families* (Marshall and Olkin, 2007). The material in this article constitutes Parts I, II, III, and IV of the book. The remaining parts relate to parametric families and to complementary topics. The topics included in this summary constitute the more novel aspects of the subject of life distributions.

As noted, the cumulative distribution has been used for the Kolmogorov–Smirnov test, the density has been studied extensively, the residual life distribution is used in life tables, the hazard rate in survival analysis. This suggests that some of these alternative representations will be useful in other contexts. For example, proportional odds rather than proportional hazards might be more suitable for some types of data.

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